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Lower bounds on zero–one matrices

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Abstract

In this paper we work with pairs of zero–one matrices whose product is the full upper-triangular zero–one matrix. We establish a lower bound for the sum of the non-zero entries in them, using elementary concepts of linear algebra. With this algebraic method we reach the best possible lower bound, since pairs of such matrices may be found whose amount of non-zero entries is exactly our lower bound.

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1. Introduction

Throughout this paper, R, U will denote $n \times m$, $m \times n$ dimensional $(0, 1)$-matrices respectively, $R = (r_{i,j})$, $U = (u_{i,j})$, with $R \times U = T$ being

$$T = (t_{i,j})_{i,j=1,\ldots,n}, \quad \text{with } t_{i,j} = \begin{cases} 1, & i \geqslant j, \\ 0, & i < j. \end{cases}$$

When necessary, we will specify the dimension of the square matrix T with a subindex, that is, T_n means the $n \times n$ dimensional matrix T.
We define

$$s_R = \sum_{i=1}^{n} \sum_{j=1}^{m} r_{i,j} \qquad s_U = \sum_{i=1}^{m} \sum_{j=1}^{n} u_{i,j}.$$

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We establish a lower bound for the function

$$\phi(n,m) = \min\{s_R + s_U\},$$

with R, U running over the matrices defined above.

Let us introduce some notation and concepts that will be useful from now on: we shall denote by $I_{m_{i,j}}$ the matrix obtained by permuting the ith and jth rows in the $m \times m$ identity I_m. For any $n \times m$ dimensional matrix A, the effect of multiplying $A \times I_{m_{i,j}}$ is to permute the ith and jth columns in A. On the other hand, if A is $m \times n$ dimensional, $I_{m_{i,j}} \times A$ is just like matrix A but with rows i, j permuting their positions. We also observe that for any i, j, m, $I_{m_{i,j}} \times I_{m_{i,j}} = I_m$. For any matrix $X = (x_{i,j})_{i=1,\dots,n,\ j=1,\dots,m}$ we shall denote:

$$X_{i,*} = (x_{i,1}, x_{i,2}, \dots, x_{i,m}),$$

$$X_{*,j} = (x_{1,j}, x_{2,j}, \dots, x_{n,j})^{\mathrm{t}}.$$

As main result, in Theorem 10 we prove that for any given pair of matrices R, U,

$$\phi(n,m) \geqslant n[\log_2 n] + [\log_2 n] + (m - 2^k) + 2 \quad (2^k \leqslant n < 2^{k+1}).$$

This is the best possible lower bound, since pairs of zero–one matrices whose product is T_n may be defined for any n, whose amount of non-zero entries is exactly $n[\log_2 n] + [\log_2 n] + (m - 2^k) + 2$.

In particular, obtaining a lower bound for ϕ is equivalent to obtaining lower bounds for the average complexity of partial sums queries and update operations over an array inside an algebraic model of computation for the study of the so called *partial sums problem* (see [1]). Within this model, these operations are represented by zero–one matrices R, U whose product is T, and the problem of finding a lower bound for the average complexity is equivalent to establish a lower bound for ϕ and divide it by the total number of operations. On the other hand, binary trees representing solutions for the partial sums problem which are proved to be optimal in terms of average complexity, have been defined in [2]. These solutions fall inside the algebraic model of computation and it is not difficult to see that when translated to the corresponding pairs of matrices the amount of non-zero entries in them is exactly our lower bound for ϕ.

We believe that our method could be use to treat some related problems (see e.g. [3–6]).

We start with an elementary algebraic remark which says us that we only have to consider the case $n \leqslant m$.

Remark 1. Let R, U be $n \times m$ and $m \times n$ dimensional respectively. Since $R \times U = T$ and T is a non-singular matrix we must have $n \leqslant m$.

2. Main results

In order to obtain the lower bound for function ϕ, we prove some preliminary results.

Lemma 2. *Given R and U, we have that for every $k \in \{1, \ldots, n\}$, there exists a non-empty set of index $A_k = \{\sigma_k^1, \ldots, \sigma_k^{i_k}\} \subset \{1, \ldots, m\}$ such that*

(i) *for every $i \in A_k$, $u_{i,k} = 1$, and for every $j > k$, $u_{i,j} = 0$*
(ii) *for every $j \in A_k$,*
  (a) *$r_{i,j} = 0$ if $i < k$,*
  (b) *there exists $j_0 \in A_k$ such that $r_{k,j_0} = 1$,*
  (c) *$r_{k,j} = 0\ \forall j \in A_k,\ j \neq j_0$.*

Proof. Without lost of generality we may assume that every column in R has a non-zero entry, since if $R_{*,j_0} = (0, \ldots, 0)^{\mathrm{t}}$ we may work with new $n \times (m-1)$, $(m-1) \times n$ matrices

$$R = \left(R_{*,1}, \ldots, R_{*,j_0-1}, R_{*,j_0+1}, \ldots, R_{*,m}\right),$$

$$U = \left(U_{1,*}, \ldots, U_{j_0-1,*}, U_{j_0+1,*}, \ldots, U_{m,*}\right).$$

Similarly, we may assume that every row in U has a non-zero entry.

On the other hand, neither two columns in R nor two rows in U are equal: since if $R_{*,j_0} = R_{*,j_1}\,(j_0 < j_1)$, we may work with

$$R = \left(R_{*,1}, \ldots, R_{*,j_0}, \ldots, R_{*,j_1-1}, R_{*,j_1+1}, \ldots, R_{*,m}\right)$$

and

$$U = (U_{1,*}, \ldots, U_{j_0-1,*}, (U_{j_0,*} + U_{j_1,*}), U_{j_0+1,*}, \ldots U_{j_1-1,*}, \\ U_{j_1+1,*}, \ldots, U_{m,*}).$$

(let us observe that from $R_{*,j_0} = R_{*,j_1}$ it follows that for every $j \in \{1, \ldots, n\}$ we have $u_{j_0,j} + u_{j_1,j} \leqslant 1$).

Meanwhile that if we have $U_{j_0,*} = U_{j_1,*}$ $(j_0 < j_1)$, it is enough to consider

$$R = (R_{*,1}, \ldots, R_{*,j_0-1}, (R_{*,j_0} + R_{*,j_1}), R_{*,j_0+1}, \ldots, R_{*,j_1-1}, \\ R_{*,j_1+1}, \ldots, R_{*,m})$$

and

$$U = (U_{1,*}, \ldots, U_{j_0,*}, \ldots U_{j_1-1,*}, U_{j_1+1,*}, \ldots, U_{m,*}).$$

Now we proceed with the proof.

Let $k \in \{1, \ldots, n\}$ and let us suppose that there is no index enjoying the condition (i). Then for every $i \in \{1, \ldots, m\}$ such that $u_{i,k} = 1$, there exists $j \in \{k+1,$

$k+2, \dots, n\}$ with $u_{i,j} = 1$. Let $i_0 \in \{1, \dots, m\}$ with $u_{i_0,k} = 1$ (such an index exists because $R \times U = T$). We know that $R_{k,*} \times U_{*,k} = 1$, so there exists z_0 such that $r_{k,z_0} = 1 = u_{z_0,k}$, but we are supposing that there exists $j > k$ with $u_{z_0,j} = 1$, so $R_{k,*} \times U_{*,j} = 1$, and this is a contradiction, so there exists a non-empty A_k verifying (i).

Now we consider two different situations: if $A_k = \{j_0\}$ then $r_{i,j_0} = 0$ if $i < k$ and $r_{k,j_0} = 1$, so (ii) is proved. Otherwise if we have $j_0 \neq j_1$, $j_0, j_1 \in A_k$, let us suppose that $r_{k,j_0} = 1 = r_{k,j_1}$; it would imply $R_{k,*} \times U_{*,k} \geqslant 2$, but this is not possible, so as $r_{i,j_1} = 0$ if $i < k$ is trivially true, we have proved (ii). □

The next corollary follows immediately from Lemma 2.

Corollary 3. *Given R, U, we have that for every $k \in \{1, \dots, n\}$,*

$$0 < \sum_{i=1}^{m} u_{i,k} \leqslant m - k + 1.$$

The following theorem tells us that we may rearrange the columns in R and the rows in U obtaining pairs of matrices with the following shape:

$$\begin{pmatrix}
\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & . & * & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & . & . & * & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & . & . & . & . & . & * & \mathbf{1} & 0 & 0 & 0 & 0\\
* & . & . & . & . & . & . & * & \mathbf{1} & 0 & 0 & 0\\
* & . & . & . & . & . & . & . & . & * & \mathbf{1} & 0\\
* & . & . & . & . & . & . & . & . & . & * & \mathbf{1}
\end{pmatrix}
\begin{pmatrix}
\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0\\
. & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0\\
. & * & \mathbf{1} & 0 & 0 & 0 & 0 & 0\\
. & . & * & \mathbf{1} & 0 & 0 & 0 & 0\\
. & . & . & \mathbf{1} & 0 & 0 & 0 & 0\\
. & . & . & \mathbf{1} & 0 & 0 & 0 & 0\\
. & . & . & * & \mathbf{1} & 0 & 0 & 0\\
. & . & . & . & * & \mathbf{1} & 0 & 0\\
. & . & . & . & . & \mathbf{1} & 0 & 0\\
. & . & . & . & . & * & \mathbf{1} & 0\\
* & * & * & * & * & * & * & \mathbf{1}
\end{pmatrix}. \qquad (*)$$

In $(*)$ $|A_1| = 1$, $|A_2| = 2$, $|A_3| = 1$, $|A_4| = 3$, $|A_5| = 1$, $|A_6| = 2$, $|A_7| = 1$ and $|A_8| = 1$.

Theorem 4. *Without lost of generality we may assume that the matrices R, U have the following properties: there exist $1 = j_1 < \cdots < j_n = m$, such that for every $k = 1, \dots, (n-1)$*

- $u_{i,k} = 1$ *if* $j_k \leqslant i < j_{k+1}$, *and* $u_{i,k} = 0$ *if* $i < j_k$,
- $u_{i,j} = 0$ *if* $j > k$,
- $r_{k,j_k} = 1$ *and* $r_{k,j} = 0$ *if* $j > j_k$.

Proof. For every $k = 1, \dots, n$, let σ_k^1 be the only index in A_k such that $r_{k,\sigma_k^1} = 1$, and let $i_k = |A_k|$.

We define

$$j_k = m - \sum_{s=k}^{n} i_s + 1 \quad \forall k = 1, \dots, n.$$

In $(*)$ the corresponding values would be: $j_1 = 1,\ j_2 = 2,\ j_3 = 4,\ j_4 = 5,\ j_5 = 8,\ j_6 = 9,\ j_7 = 11$ and $j_8 = 12$.

In order to see that $j_1 = 1$, it is enough to observe that $\sum_{s=1}^{n} i_s = m$, since otherwise, if $\sum_{s=1}^{n} i_s < m$, there would necessarily exist $z_0 \in \{1, \dots, n\}$ such that $R_{z_0,*} = (0, \dots, 0)^{\mathrm{t}}$, and we assume that this is not possible.

On the other hand, $j_n = m$ is equivalent to proving that $i_n = 1$; let us suppose that $i_n > 1$, it would imply that there exist $z_1,\ z_2 \in A_n,\ z_1 \neq z_2$ such that $u_{z_1,n} = 1 = u_{z_2,n}$, but, as $R \times U = T$, it means that either R_{*,z_1} or R_{*,z_2} is identically zero, and we are supposing this is not possible.

Let us call $M_k = I_{\sigma_k^{i_k}, j_{k+1}-1} \times \cdots \times I_{\sigma_k^1, j_k}$ (let us observe that it implies $M_k^{-1} = I_{\sigma_k^1, j_k} \times \cdots \times I_{\sigma_k^{i_k}, j_{k+1}-1}$).

It is enough to realize that if we consider new matrices R, U defined as:

$$R' = (\cdots((R \times M_n) \times M_{n-1}) \times M_1),$$

$$U' = (M_1^{-1} \times (\cdots(M_{n-1}^{-1} \times (M_n^{-1} \times U))\cdots)$$

then we have that $R' \times U' = T$, $m_{R'} = m_R$, $m_{U'} = m_U$, $s_{R'} = s_R$, $s_{U'} = s_U$. On the other hand, it is clear that the new matrices R', U' enjoy the required properties. □

The following corollaries are trivially true.

Corollary 5. *If $m = n$, then we may assume that R, U are upper triangular.*

Corollary 6

$$\sum_{k=1}^{n} \sum_{s=j_k+1}^{j_{k+1}-1} u_{s,k} + \sum_{k=1}^{n} (r_{k,j_k} + u_{j_k,k}) = 2n + (m - n).$$

Remark 7. Notice that if $m = n$, we have just obtained the sum of the main diagonals.

By the moment we will consider that R, U are $n \times m$, $m \times n$ dimensional matrices with $m \geqslant n$ and $n = 2^k$ for a certain natural k. Let us observe that in this case, the Corollary 6 above says

$$\sum_{k=1}^{n} \sum_{s=j_k+1}^{j_{k+1}-1} u_{s,k} + \sum_{k=1}^{n} (r_{k,j_k} + u_{j_k,k}) = 22^k + (m - n).$$

The aim of the following two lemmas is to establish a lower bound for the sum of the entries in R and U. We start counting the *subdiagonal* ones.

Lemma 8. *Given R, U we have*

$$\sum_{k=1}^{2^k-1}\left[\sum_{j=j_k}^{j_{k+1}-1} r_{k+1,j} + u_{j_{k+1},k}\right] = 2^k - 1 = 2^k - 2^1 + 1.$$

Proof. The result follows from

$$R_{k+1,*} \times U_{*,k} = 1 \quad \forall k = 1, \ldots, n-1$$

and the fact that, by Theorem 4, we have that

$$r_{k+1,z} = \begin{cases} 1 & \text{if } z = j_{k+1}, \\ 0 & \text{if } z > j_{k+1}, \end{cases}$$

$$u_{z,k} = \begin{cases} 1 & \text{if } j_k \leqslant z < j_{k+1}, \\ 0 & \text{if } z < j_k \end{cases}$$

for every $k = 1, \ldots, n-1$. □

Lemma 9. *Given R, U, for every $i = 0, \ldots, k-2$,*

$$\sum_{z=0}^{2^k-2^{k-i}}\left(\sum_{s=j_{z+1}}^{j_{z+1+2^{k-(i+1)}}-1} r_{2^{k-i}+z,s} + \sum_{s=j_{z+1+2^{k-(i+1)}}}^{j_{z+2^{k-i}}} u_{s,z+1}\right) \geqslant 2^k - 2^{k-i} + 1.$$

Proof. It is enough to prove that for every $z = 0, \ldots, 2^k - 2^{k-i}$, we have

$$\sum_{s=j_{z+1}}^{j_{z+1+2^{k-(i+1)}}-1} r_{2^{k-i}+z,s} + \sum_{s=j_{z+1+2^{k-(i+1)}}}^{j_{z+2^{k-i}}} u_{s,z+1} \geqslant 1. \tag{1}$$

Let us suppose that there exists $t \in \{0, \ldots, 2^k - 2^{k-i}\}$ such that

$$\sum_{s=j_{t+1}}^{j_{t+1+2^{k-(i+1)}}-1} r_{2^{k-i}+t,s} = 0.$$

Then by Theorem 4 and the fact that $R_{2^{k-i}+t,*} \times U_{*,t+1} = 1$, it follows that $\sum_{s=j_{z+1+2^{k-(i+1)}}}^{j_{z+2^{k-i}}} u_{s,z+1} \geqslant 1$. We still have to prove that each entry in the matrices appears at most once in (1).

First we prove it for any row in R: let us suppose that there exist $v, w \in \{0, \ldots, k-2\}$, $z_v \in \{0, \ldots, 2^k - 2^{k-v}\}$, $z_w \in \{0, \ldots, 2^k - 2^{k-w}\}$ such that

$$2^{k-v} + z_v = 2^{k-w} + z_w, \quad v < w.$$

It follows that

$$z_w - z_v = 2^{k-v} - 2^{k-w}. \tag{2}$$

We have to prove that $j_{z_v+1+2^{k-(v+1)}} \leqslant j_{z_w+1}$ but, as $x < y \Longrightarrow j_x < j_y$, it is enough to prove $z_v + 1 + 2^{k-(v+1)} \leqslant z_w + 1$ or equivalently $z_v + 2^{k-(v+1)} \leqslant z_w$.

From the fact that $v < w$, it follows that $2^w \geqslant 2^{w-1} + 2^v$ and therefore $2^{k+w} - 2^{k+v} \geqslant 2^{k+w-1}$.

Dividing by 2^{v+w}, we have that $2^{k-v} - 2^{k-w} \geqslant 2^{k-(v+1)}$, hence, by (2), it implies $z_v + 2^{k-(v+1)} \leqslant z_w$.

Now we prove it for any column in U: let $v, w \in \{0, \dots, k-2\}$, $v < w$. We have to prove that

$$j_{z+2^{k-w}} < j_{z+1+2^{k-(v+1)}},$$

but, as j is strictly non-decreasing, it is enough to prove

$$z + 2^{k-w} < z + 1 + 2^{k-(v+1)}$$

or equivalently,

$$2^{k-w} - 2^{k-(v+1)} \leqslant 0.$$

Multiplying by 2^{w+v-k} we have $2^v - 2^{w-1} \leqslant 0$, and this is trivially true because $v \leqslant w - 1$. □

Now we are ready to obtain the lower bound we are looking for.

Theorem 10. *Given R, U, and a natural number k such that $n = 2^k$,*

$$\phi(n, m) \geqslant n \log_2 n + \log_2 n + (m - 2^k + 2).$$

Proof. The theorem is a direct consequence of Corollary 6, Lemmas 8 and 9. □

Let us observe that in case $2^k < n < 2^{k+1}$ for a certain natural number k, the lower bound that we obtain is $n[\log_2 n] + [\log_2 n] + (m - 2^k) + 2$, as there are a few things that change from the proof above. In Lemma 9 it should say

$$\sum_{z=0}^{n-2^{k-i}} \left(\sum_{s=j_{z+1}}^{j_{z+1+2^{k-(i+1)}}-1} r_{2^{k-i}+z,s} + \sum_{s=j_{z+1+2^{k-(i+1)}}}^{j_{z+2^{k-i}}} u_{s,z+1} \right) \geqslant n - 2^{k-i} + 1.$$

Lemma 8 should say

$$\sum_{k=1}^{n-1} \left[\sum_{j=j_k}^{j_{k+1}-1} r_{k+1,j} + u_{j_{k+1},k} \right] = n - 1 = n - 2^1 + 1.$$

On the other hand, we observe that for every $z = 1, \dots, (n - 2^k)$, $R_{2^k+z,*} \times U_{*,z+1} = 1$, so $\sum_{z=1}^{n-2^k} [\sum_{s=1}^{j_{z+1+2^{k-1}}-1} (r_{2^k+z,s} + \sum_{l=j_{z+2^{k-1}+1}}^{j_{z+2^k}} u_{l,(z+1)})] \geqslant (n - 2^k)$ (let us observe that $j_{z+1+2^{k-1}} - 1 \geqslant j_{z+2^{k-1}}$).

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